

THE MONSTER

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OBJECTIVE

We will educate the reader on the project which motivated the discovery of sporadic groups, what they are and familiarize them with some interesting properties of the largest sporadic group, the monster. The reader should be familiar with the notion of a simple group.

HISTORY

Simple groups can be thought of as the "building blocks" of group theory, so classifying all of them was thought of by many as "solving" group theory.

In 1972, Daniel Gorenstein launched the goal of classifying all finite simple groups (up to isomorphism). By the end of the project in 2008, it had been discovered that nearly all finite simple groups can be classified into one of 18 infinite families.

However there are some exceptions. There exist 26 "insubordinate" finite simple groups which can not be classified into these 18 families.

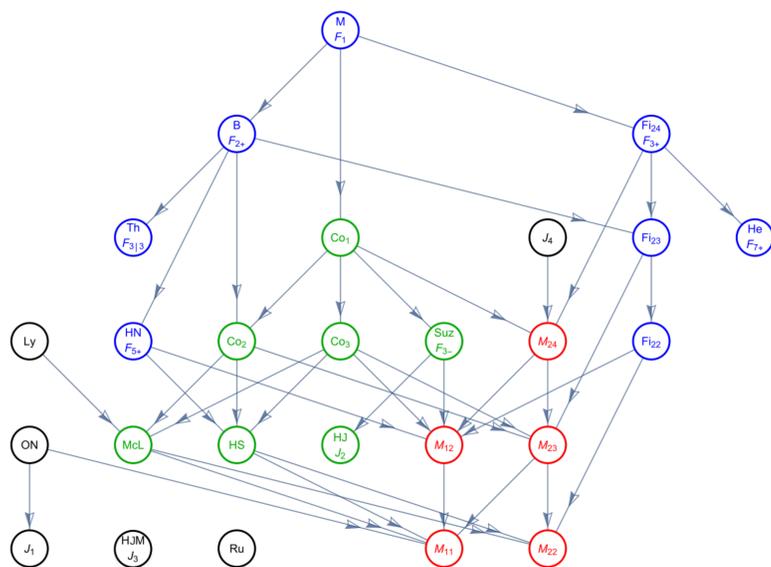
These exceptions are known as the sporadic groups.

Around 1973 Bernd Fischer and Robert Griess predicted that the simple group now known as the Monster might exist, based off of the then unproven existence of a simple group now known as the Baby Monster.

From this prediction, many properties of the Monster were discovered, including its order and character table over the following years.

It was then proven that 20 of the sporadic groups were either subgroups or subquotients of the Monster.

In 1980, Griess provided a construction of the Monster, proving its existence.

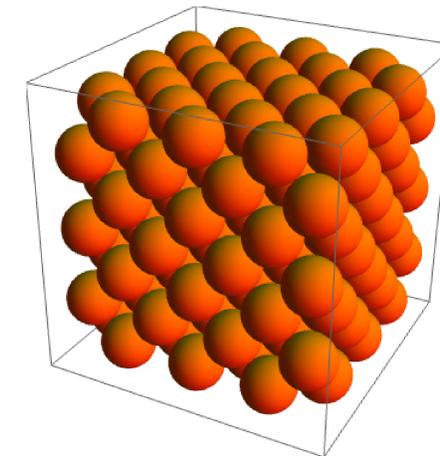


THE MONSTER REPRESENTATION

Representation theory describes a way to take group elements and rewrite them as matrices, with matrix multiplication. We can use this to write the group of non-commutative bilinear transformations as a group of matrices.

The smallest nontrivial representation of the Monster acts on a vector space of dimension 196883, and the group can be generated by 2 matrices of 196883×196883 .

The elements of the Monster Group have been described as the intertwining of symmetries between the 24th dimension of the Golay Code, the space of 24×24 symmetric matrices, and the 24th dimension of the Leech Lattice.



The order of the Monster group: $|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \approx 8.08 \times 10^{53}$

MONSTROUS MOONSHINE

Mathematicians had already computed many of the Monster's representation matrices over \mathbb{C} . They also wanted to understand its representations over modular fields, using modular arithmetic. In the late 1970s, John McKay noticed a striking numerical coincidence while studying modular functions arising in Galois theory and elliptic curves.

The elliptic modular function is

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots = \sum c(n)q^n, \quad q = e^{2\pi i\tau}.$$

McKay observed that the first nontrivial coefficient satisfies

$$196884 = 1 + 196883,$$

where 196883 is the dimension of the Monster's smallest nontrivial representation (see above).

At first, this was dismissed as a coincidence. But John Thompson soon noticed that several of the next coefficients of $j(\tau)$ also decompose as simple linear combinations of the dimensions of irreducible Monster representations. For example, the Monster also has a 21296876-dimensional irreducible representation, and

$$21493760 = 21296876 + 196883 + 1.$$

Main Result (Moonshine): McKay and Thompson conjectured that there exists an infinite-dimensional graded vector space

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

such that $\dim V_n = c(n)$, where $c(n)$ is the q^n coefficient of $j(\tau)$. Each V_n carries an action of the Monster, and decomposes into irreducible Monster representations.

V was later proven to exist, and is now known as the *Monster vertex algebra*, and the Monster is the group of its automorphisms from $V \rightarrow V$.

CONCLUSIONS AND REFERENCES

In conclusion, the Monster is extremely large, and its representation matrices are complicated to visualize, to say the least. However, as the largest sporadic group, the Monster provides insight into how grand the sporadic groups really are, and how intricate group theory can be.

References:

- <https://www.ams.org/notices/200209/what-is.pdf>
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- <https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-14/issue-1/Classifying-the-finite-simple-groups/bams/1183552783.pdf>